Spatial Kepler Problem

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Developed result of following paper to dimension 3.

THE CONLEY-ZEHNDER INDICES OF THE ROTATING KEPLER PROBLEM

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ABSTRACT. We determine the Conley-Zehnder indices of all periodic orbits of the rotating Kepler problem for energies below the critical Jacobi energy. Consequently, we show the universal cover of the bounded component of the regularized energy hypersurface is dynamically convex. Moreover, in the universal cover there is always precisely one periodic orbit with Conley-Zehnder index 3, namely the lift of the doubly covered retrograde circular orbit.

1. INTRODUCTION

The Kepler problem in rotating coordinates arises as the limit of the planar circular restricted 3-body problem when the mass of one of the primaries goes to zero, and hence serves as an approximation of the restricted planar 3-body problem for a small mass parameter. The ultimate goal is to study the dynamics of the 3-body problem using finite energy foliations. One essential ingredient is the so-called Conley-Zehnder index of a periodic orbit. These indices play a central role in the theory of finite energy foliations, symplectic field theory, Fukaya A_∞-categories, and various Floer theories.

Based on the joint work with Beomjun Sohn and Sunghae Cho.

Main Result

1. Describing the moduli space of spatial Kepler orbit.
2. Computation of the Conley-Zehnder index of periodic orbits of rotating Kepler problem.

Contents

Spatial Kepler Problem

Three Laws of Kepler

Hamiltonian : **Kepler energy** $E : T^*(\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R}$

$$E(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}$$

E describes the motion of an object under the gravitational force of a mass at the origin.

1. The solutions are conic sections with one focus at the origin. If $E < 0$, every orbit is an ellipse.
2. The areal velocity $\dot{S} = r^2\dot{\theta}/2$ is constant.
3. The period τ of solution satisfies $\tau^2 = -\pi^2/2E^3$. τ only depends on the Kepler energy.

Kepler Orbit

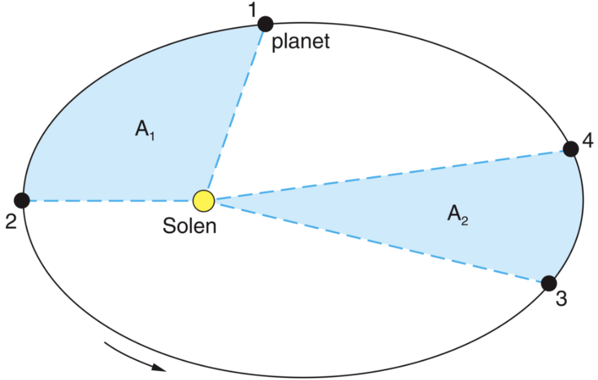


Figure 1: Illustration of Kepler orbit [1]

[1] https://snl.no/Keplers_problem

Invariant - Angular Momentum

E has $SO(3)$-symmetry $\Rightarrow$ **angular momentum** is an invariant.

$$L = (L_1, L_2, L_3) = q \times p$$

Invariance of $L \Leftrightarrow$ invariance of the areal velocity.

($L = r^2\dot{\theta}$ in polar coordinates)

L is orthogonal to the plane which the orbit is contained in.

$\Rightarrow L$ specifies the plane.

Also, L specifies the direction of rotation.

Ex. For planar orbit, $L_1 = L_2 = 0$ and L_3 can have both signs.

L_3 positive / negative $\Rightarrow$ counterclockwise / clockwise on q_1q_2-plane

Invariant - Angular Momentum

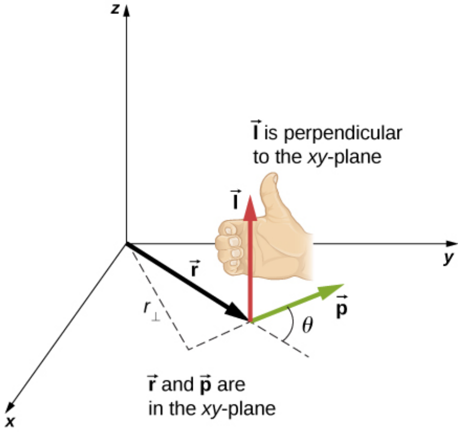


Figure 2: Angular momentum [2]

[2] W. Moebs, S. Ling, J. Sanny "General Physics Using Calculus I"

Invariant - Laplace-Runge-Lenz Vector

Laplace-Runge-Lenz vector (LRL vector) is also an invariant.

$$A = p \times L - \frac{q}{|q|}$$

Direction of A = Direction of major axis

Eccentricity : $\varepsilon^2 = A^2 = 2EL^2 + 1 \Rightarrow A = 0$ if the orbit is circular.

$$\begin{aligned} |A|^2 &= |p \times L|^2 - \frac{2}{|q|}\langle p \times L, q\rangle + 1 = |p|^2|L|^2 - \frac{2}{|q|}\langle q \times p, L\rangle + 1 \\ &= |p|^2|L|^2 - \frac{2}{|q|}|L|^2 + 1 = 2\left(|p|^2 - \frac{1}{|q|}\right)|L|^2 + 1 = 2EL^2 + 1. \end{aligned}$$

Corresponding symmetry is called *hidden symmetry*. (Appendix 1)

Invariant - Laplace-Runge-Lenz Vector

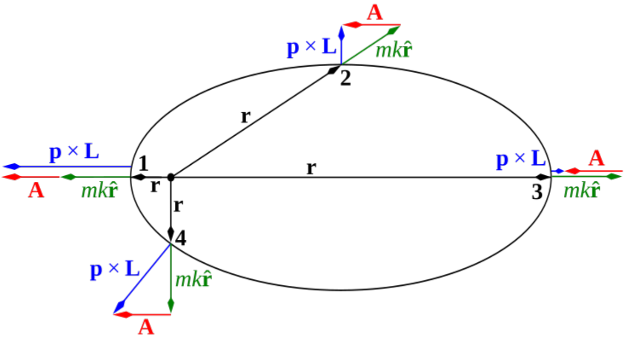


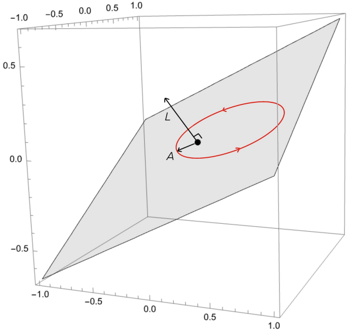
Figure 3: LRL vector [3]

[3] https://en.wikipedia.org/wiki/Laplace-Runge-Lenz_vector

Kepler Orbit

On $L \cdot q = 0$, the Kepler orbit is given in the polar coordinate by

$$r = \frac{|L|^2}{1 + |A|\cos(\theta - g)} \text{ (g is determined by the direction of A).}$$



In particular, E, L and A determine the Kepler orbit.

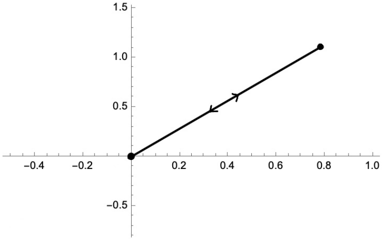
Moser Regularization

Recall.

Moser regularization embeds the level set $E^{-1}(E_0)$ of the Kepler problem into the geodesic flow on $T^*S^3_r$ where $r = \sqrt{-2E_0}$.

$\Rightarrow$ Compactification of the energy level set.

The **collision orbits** (great circles passing the point at infinity) are added.



This is special case of elliptic orbit with $\varepsilon = |A| = 1$, $L = 0$.

Rotating Kepler Problem

Motivation

Motivation : **Restricted circular three-body problem**

Motion of a massless body under the gravitational force of two objects with mass ratio μ, and assume the motions of two bodies are circular. Corresponding Hamiltonian is time-dependent.

$$E_t(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - m(t)|} - \frac{1-\mu}{|q - e(t)|},$$
$$e(t) = -\mu(\cos t, -\sin t, 0),\ m(t) = (1-\mu)(\cos t, -\sin t, 0)$$

In rotating frame, the Hamiltonian is autonomous (time-independent).

$$H = \frac{1}{2}|p|^2 - \frac{\mu}{|q - (1-\mu)|} - \frac{1-\mu}{|q-\mu|} + (q_1 p_2 - q_2 p_1)$$

Rotating Kepler problem is a limit case, $\mu = 0$.

Motivation

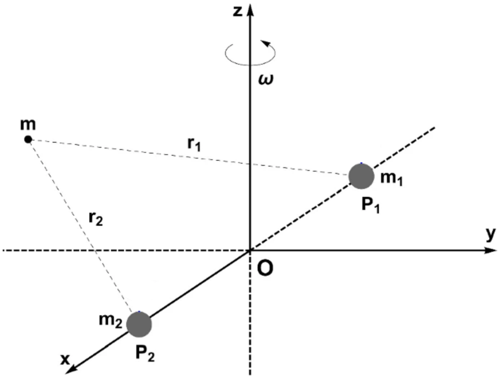


Figure 4: Restricted circular three-body problem[4]

[4]H. Alrebdi, F.Dubeibe, K,Papadakis, E.Zotos "Equilibrium dynamics of a circular restricted three-body problem with Kerr-like primaries"

Rotating Kepler Problem

Rotating Kepler problem : Kepler problem with rotating frame

$$H = E + L_3 = \frac{1}{2}|p|^2 - \frac{1}{|q|} + (q_1 p_2 - q_2 p_1)$$

We perform Moser regularization on the compact component $H^{-1}(c)$.

The Hill's region has a compact component if $c < -3/2$. (Appendix 2)

The regularized system is a Finsler geodesic flow on T^*S^3.

Periodic Orbits

Three types of periodic orbits.

1. **Planar circular orbits**, nondegenerate for generic c.
2. **Vertical collision orbits**, nondegenerate for generic c.
3. **Degenerate elliptic orbits**.

Planar Circular Orbits

1. Condition of c, E to be circular

$\{E, L_3\} = 0 \Rightarrow Fl_t^H = Fl_t^E \circ Fl_t^{L_3}$

$Fl_t^{L_3}$ is a rotation of period 2π along q_3- and p_3-axes.

Planar circular orbit composed with φ^{L_3} is always periodic.

Circular condition: $\varepsilon^2 = 2EL_3^2 + 1 = 2E(c-E)^2 + 1 = 0$, or

$$c_\pm = E \pm \frac{1}{\sqrt{-2E}}$$

Planar Circular Orbits

1. Condition of c, E to be circular

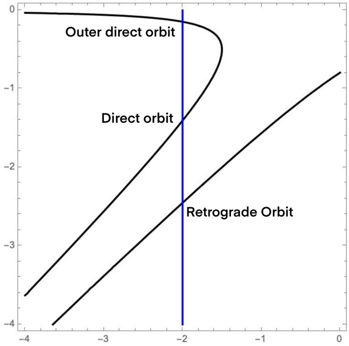


Figure 5: Graph of $2E(c-E)^2+1=0$ in (c,E)-plane.

Planar Circular Orbits

2. Retrograde and Direct Orbits

For fixed $c < -3/2$, there are exactly 3 planar circular orbits.

Retrograde orbit γ_+: $L_3 = 1/\sqrt{-2E}$, $A = 0$

Rotates counterclockwise.

Direct orbit γ_-: $L_3 = -1/\sqrt{-2E}$, $A = 0$

Rotates clockwise.

The rest one (outer direct orbit) lies on the unbounded component of the Hill's region, and not of our interest.

Note. The Kepler energy E characterizes $\gamma_\pm$,

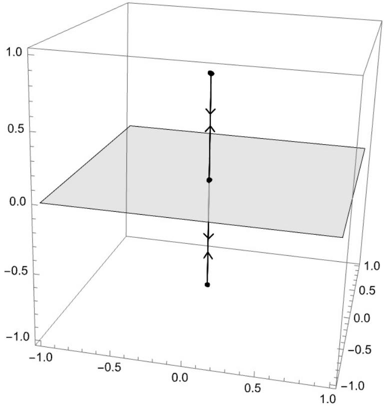
i.e. there exists only one retrograde orbit and only one direct orbit with a given Kepler energy E.

Vertical Collision Orbits

Vertical collision orbits $\gamma_{c_\pm}$: $L = 0$, $A_3 = \mp 1$, $c = E$.

These are not effected by φ^{L_3}, since $q_1 = q_2 = p_1 = p_2 = 0$.

$\Rightarrow$ Periodic for every energy level c.



Degenerate Elliptic Orbits

To make other orbits periodic, the period must be rational multiple of 2π.

We have $\tau = 2\pi/(-2E)^{3/2}$, which implies that

$$k\tau = \frac{2k\pi}{(-2E)^{3/2}} = 2l\pi \;\Rightarrow\; E_{k,l} = -\frac{1}{2}\left(\frac{k}{l}\right)^{2/3}$$

For given c, only the elliptic orbit with Kepler energy $E_{k,l}$ and angular momentum $L_3 = c - E_{k,l}$ can be periodic.

Possible value for L_3 ($\varepsilon^2 = 2EL_3^2 + 1 \geq 0$):

$$-\frac{1}{\sqrt{-2E_{k,l}}} \leq L_3 \leq \frac{1}{\sqrt{-2E_{k,l}}}$$

The equality holds for the planar circular orbits.

Degenerate Elliptic Orbits

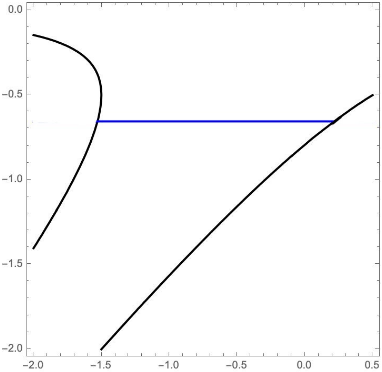


Figure 6: $E_{3,2}$, drawn with $2E(c-E)^2+1=0$.

1. The endpoints are $\gamma_\pm$.
2. Each interior point is degenerate family of elliptic orbits.

Degenerate Elliptic Orbits

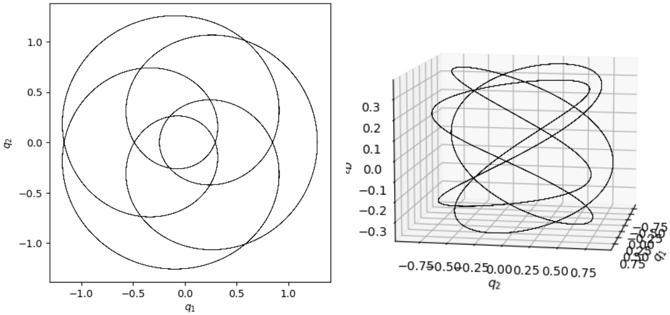


Figure 7: Illustration of periodic orbits on a plane and space [5]

Such orbits always appear with S^3-family. (will be explained)

Intuition : $SO(3)$-rotation, possibly 2 rotating directions.

[5] Thank you for nice picture, Chankyu Joung.

Moduli Space of Kepler Orbits

Parametrization of the Moduli Space

Important relations

1. $\langle L, A \rangle = 0.$
2. $\varepsilon^2 = A^2 = 2EL^2 + 1$
3. $||\sqrt{-2E}L \pm A||^2 = 1$
4. E, L, and A characterizes the Kepler orbit.

Denote $x = \sqrt{-2E}L - A$, $y = \sqrt{-2E}L + A$.

The moduli space of the Kepler orbits with Kepler energy E is

$$\mathcal{M}_E = \{(x, y) \,:\, |x|^2 = |y|^2 = 1\} \simeq S^2 \times S^2$$

Note. (Space of unit geodesics of S^3) $= ST^*S^3/S^1 \simeq S^2 \times S^2$.

Properties of $\mathcal{M}_E$

Under parametrization $(\sqrt{-2E}L - A, \sqrt{-2E}L + A)$ of $\mathcal{M}_E$,

1. Eccentricity $= |A| = |x - y|/2$
2. Circular orbits $= \{A = 0\} = \{x = y\} \simeq S^2$
3. Collision orbits $= \{L = 0\} = \{x = -y\} \simeq S^2$
4. Planar orbits $= \{L_1 = L_2 = A_3 = 0\} \simeq S^2$ (Appendix 3)
5. Retrograde and Direct orbits $\gamma_\pm = \{((0,0,\pm 1),(0,0,\pm 1))\}$
6. Vertical collision orbits $\gamma_{c_\pm} = \{((0,0,\pm 1),(0,0,\mp 1))\}$

L_3 as a Morse Function on $\mathcal{M}_E$

$L_3 = \frac{x_3+y_3}{2\sqrt{-2E}}$ is a Morse function on $\mathcal{M}_E \simeq S^2 \times S^2$ such that

1. γ_- is the unique minimum of Morse index 0. $(L_3 = -1/\sqrt{-2E}.)$
2. $\gamma_{c\pm}$ are critical points of Morse index 2. $(L_3 = 0)$
3. γ_+ is the unique maximum of Morse index 4. $(L_3 = 1/\sqrt{-2E}.)$

Let $c - E = C$, where $H = c$.

$H^{-1}(c)$ contains $L_3^{-1}(C) \simeq S^3$ for each $E_{k,l}$ if $C \neq \pm 1/\sqrt{-2E}, 0$.
(handle attachment)

$L_3^{-1}(0)$ is homeomorphic to the suspension of T^2 (not a manifold).

$\Rightarrow$ Degenerate S^3-families of orbits are contained in
the set of periodic orbits of H as a component.

Illustration of $\mathcal{M}_E$

Similarly, $A_3 = \frac{y_3 - x_3}{2}$ is a Morse function on $\mathcal{M}_E$.

The image and fibers of the map $(L_3, A_3) : \mathcal{M}_E \to \mathbb{R}^2$ is

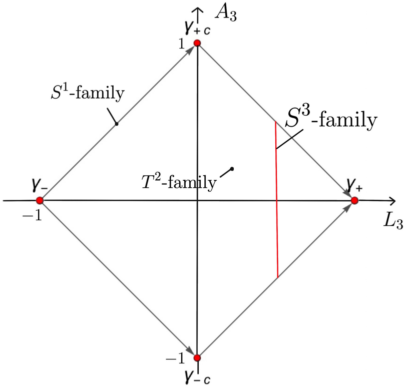
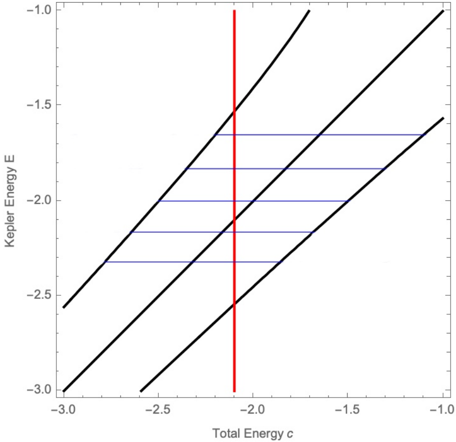


Figure 8: Toric-style illustration of $\mathcal{M}_E$.

Orbits in a Energy Level Set



For energy level $c \neq E_{k,l}$, $H^{-1}(c)$ (red line) contains

1. Retrograde and direct orbits with Kepler energy $E = E_{\pm}$.
2. Two vertical collision orbits with Kepler energy $E = c$.
3. S^3-family of elliptic orbits for each $E_+ < E_{k,l} < E_-$.

Conley-Zehnder Index of Kepler Orbits

Goal

1. Compute Conley-Zehnder indices of nondegenerate orbits, $\gamma_{\pm}$, $\gamma_{c_{\pm}}$.
2. Compare the result with (S^1-equivariant) symplectic homology.
3. For degenerate orbits, we use Morse-Bott spectral sequence.

Summary of the Result

Orbits	**Initial Index** ($c \ll -3/2$)	**Index Change**
Retrograde γ_+^N	$4N-2$	-4 at $E_{N-k,k}$ for $k=1,\dots,N-1$
Direct γ_-^N	$4N+2$	$+4$ at $E_{N+k,k}$ for $k=1,2,\dots$
Vertical Collisions $\gamma_{c\pm}^N$	$4N$	No change
Degenerate S^3-family	$4k-\frac{1}{2}$	Appears at $E_{k,l}$

Table 1: Index changes for different orbit types.

Robbin-Salamon Index

$\Psi : [0, \tau] \to Sp(2n)$: path of symplectic matrices

Crossing: t such that $\det(\Psi(t) - \mathrm{Id}) = 0$

Crossing Form: $Q_t(v, v) = \omega(v, \dot{\Psi}(t)v)$

Robbin-Salamon index is a half-integer

$$\mu_{RS}(\Psi) = \frac{1}{2}\mathrm{Sign}Q_0 + \sum_{t\,:\,\text{crossing}} \mathrm{Sign}Q_t + \frac{1}{2}\mathrm{Sign}Q_\tau$$

1. μ_{RS} is invariant under homotopy.
2. If $\Psi_3(t) = \Psi_1(t)\Psi_2(t)$, $\mu_{RS}(\Psi_3) = \mu_{RS}(\Psi_1) + \mu_{RS}(\Psi_2)$.

Note. Possible other conventions, but will be the same.

Conley-Zehnder Index

γ: nondegenerate contractible periodic Reeb orbit of $(Y, \ker \alpha)$

$A : \gamma^*\xi \to [0, \tau] \times \mathbb{R}^{2n}$: trivialization of ξ, which can be extended to a capping disk.

Conley-Zehnder index of γ is RS-index of linearized Reeb flow,

$$\Psi(t) = A(t) dFl_t^R|_\xi A(0)^{-1} \in Sp(2n)$$

Note. If $Y = H^{-1}(c)$ is given by a regular level set of contact type, μ_{CZ} of the Reeb orbit on Y, a reparametrization of Hamiltonian orbit, can be computed by linearized Hamiltonian flow restricted to ξ.
(See Appendix 4)

(Very Simple) Symplectic Homology

More explanations in Appendix 4.

W : Liouville domain, so $\partial W = Y$ is a contact manifold.

$SH_*^+(W)$: two generators for each periodic Reeb orbit of Y.

The degree is given by $\mu_{CZ}(\gamma)$ and $\mu_{CZ}(\gamma) + 1$.

Fact. (Viterbo) $SH_*(T^*M)$ is isomorphic to $H_*(\mathcal{L}M)$.

SH_*^+ is filtered by the periods (= symplectic action) of Reeb orbits.

$SH_*^{S^1,+}(W)$: one generator for each periodic Reeb orbit of Y.

The degree is given by $\mu_{CZ}(\gamma)$.

$$SH_*^{S^1,+}(T^*S^3) \simeq \begin{cases} \mathbb{Z}_2 & * = 2 \\ \mathbb{Z}_2^2 & * = 2k \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

(Very Simple) Morse-Bott Spectral Sequence

Case : Reeb orbits are *nicely* degenerate (Morse-Bott condition), and the degenerate orbits with the same period form a submanifold Σ.

Theorem

There exists a spectral sequence converging to $SH^{+,S^1}(W)$ *whose* E^1*-page is given by*

$$E^1_{pq}(SH^{S^1,+}) = \begin{cases} \bigoplus_{\Sigma \in C(p)} H^{S^1}_{p+q-\mathrm{shift}(\Sigma)}(\Sigma) & p > 0 \\ 0 & p \leq 0 \end{cases}$$

where $\mathrm{shift}(\Sigma) = \mu_{RS}(\Sigma) - \frac{1}{2}\dim \Sigma / S^1$.

Conley-Zehnder Index of Planar Circular Orbits

1. Parametrization of the Orbits

Cylindrical coordinate

$$(q_1, q_2, q_3) = (r\cos\theta, r\sin\theta, z)$$
$$(p_1, p_2, p_3) = (p_r\cos\theta - \frac{p_\theta}{r}\sin\theta, p_r\sin\theta + \frac{p_\theta}{r}\cos\theta, p_z)$$

Hamiltonian vector field

$$X_H = p_r\partial_r + \left(\frac{p_\theta}{r^2} + 1\right)\partial_\theta + p_z\partial_z + \left(\frac{p_\theta^2}{r^3} - \frac{r}{(r^2+z^2)^{3/2}}\right)\partial_{p_r} - \frac{z}{(r^2+z^2)^{3/2}}\partial_{p_z}$$

Imposing $r = r_0$ (circular) and $z = p_z = 0$ (planar).

$\Rightarrow p_r = 0$ and $r = p_\theta^2$, so for $\omega_0 = \pm\sqrt{r_0} = \pm 1/\sqrt{-2E}$,

$$X_H = \left(\frac{1}{\omega_0^3 + 1}\right)\partial_\theta$$

Conley-Zehnder Index of Planar Circular Orbits

1. Parametrization of the Orbits

Planar circular orbits are given by

$$\begin{pmatrix} r(t) \\ \theta(t) \\ z(t) \\ p_r(t) \\ p_\theta(t) \\ p_z(t) \end{pmatrix} = \begin{pmatrix} \omega_0^2 \\ \left(\frac{1}{\omega_0^3} + 1\right) t \\ 0 \\ 0 \\ \omega_0 \\ 0 \end{pmatrix}$$

Periods are given by

$$\tau_\pm = \pm \frac{2\pi}{1/\omega_0^3 + 1} = \frac{2\pi}{(-2E)^{3/2} \pm 1}.$$

Conley-Zehnder Index of Planar Circular Orbits
2. Linearized Flow

Linearized Hamiltonian flow (differentiate X_H)

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ -2/\omega_0^5 & 0 & 0 & 0 & 1/\omega_0^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/\omega_0^6 & 0 & 0 & 0 & 2/\omega_0^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\omega_0^6 & 0 & 0 & 0 \end{pmatrix}$$

Conley-Zehnder Index of Planar Circular Orbits

2. Linearized Flow

Symplectic frame of $\xi = \ker(dH) \cap \ker(-qdp)$

$$X_1 = \partial_\theta + \frac{1}{\omega_0}\partial_{p_r}, \quad X_2 = \omega_0 \partial_r$$
$$X_3 = \partial_{p_z}, \qquad X_4 = \partial_z$$

Under this basis, we have

$$\mathbf{L} = \begin{pmatrix} 0 & -1/\omega_0^4 & 0 & 0 \\ 1/\omega_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\omega_0^6 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Conley-Zehnder Index of Planar Circular Orbits

3. Crossing Forms

By integration,

$$\Psi_H(t) = \begin{pmatrix} \cos\frac{t}{\omega_0^3} & -\frac{1}{\omega_0}\sin\frac{t}{\omega_0^3} & 0 & 0 \\ \omega_0 \sin\frac{t}{\omega_0^3} & \cos\frac{t}{\omega_0^3} & 0 & 0 \\ 0 & 0 & \cos\frac{t}{\omega_0^3} & -\frac{1}{\omega_0^3}\sin\frac{t}{\omega_0^3} \\ 0 & 0 & \omega_0^3 \sin\frac{t}{\omega_0^3} & \cos\frac{t}{\omega_0^3} \end{pmatrix}$$

Crossings occurs at $2\omega_0^3\pi\mathbb{Z}$ and crossing form is

$$\Omega\dot{\Psi}_H(t) = \Omega\mathbf{L} = \begin{pmatrix} 1/\omega_0^2 & 0 & 0 & 0 \\ 0 & 1/\omega_0^4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\omega_0^6 \end{pmatrix}$$

which has signature 4. (Ω is the matrix represents the symplectic form.)

Conley-Zehnder Index of Planar Circular Orbits

4. The Formula

Theorem

Let $\gamma_\pm$ be the retrograde and direct orbits of Kepler energy E where $E \neq E_{k,l}$ for any k, l. Then $\gamma_\pm$ and their multiple covers are non-degenerate. The Conley-Zehnder index of N-th iterate of $\gamma_\pm$ is

$$\begin{aligned}\mu_{CZ}(\gamma_\pm^N) &= 2 + 4\max\left\{n \in \mathbb{Z}_{>0} : 2\pi\omega_0^3 n < N\tau_\pm\right\} \\ &= 2 + 4\max\left\{n \in \mathbb{Z}_{>0} : n < N\frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}\right\} \\ &= 2 + 4\left\lfloor N\frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}\right\rfloor\end{aligned}$$

Note. This is twice the index of circular orbits of the planar problem.

Conley-Zehnder Index of Planar Circular Orbits

5. Description by Kepler Energy E

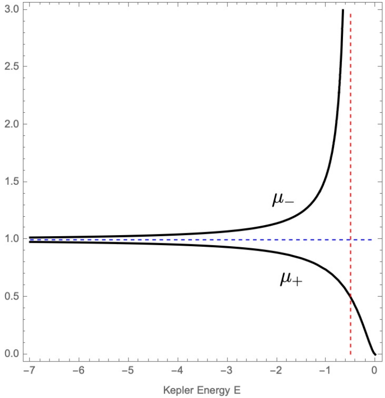


Figure 9: Graph of $\mu_\pm = \frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}$.

The index of γ_+^N decreases by 4, while γ_-^N increases by 4, whenever $\mu_\pm$ touches $k/N \Leftrightarrow E = E_{N-k,k}$ or $E = E_{N+k,k}$.

Conley-Zehnder Index of Planar Circular Orbits

5. Description by Kepler Energy E

Theorem

The index of $\gamma_{\pm}^{N}$ with Kepler energy E is given as following.

$$\mu_{CZ}(\gamma_{+}^{N}) = \begin{cases} 4N-2 & \text{if } E < E_{N-1,1}, \\ 4(N-k)-2 & \begin{array}{c}\text{if } E_{N-k,k} < E < E_{N-k-1,k+1} \\ \text{for } k = 1,2,\ldots,N-2,\end{array} \\ 2 & \text{if } E > E_{1,N-1}, \end{cases}$$

$$\mu_{CZ}(\gamma_{-}^{N}) = \begin{cases} 4N+2 & \text{if } E < E_{N+1,1}, \\ 4(N+k)+2 & \begin{array}{c}\text{if } E_{N+k,k} < E < E_{N+k+1,k+1} \\ \text{for } k = 1,2,\ldots.\end{array} \end{cases}$$

Conley-Zehnder Index of Vertical Collision Orbits

1. Decomposition of the Index

γ_c : Vertical collision orbits

K_E : Regularized (non-rotating) Kepler Hamiltonian

Ψ_{K_E} : Linearized Hamiltonian flow of K_E.

Ψ_{L_3} : Linearized Hamiltonian flow of L_3.

Lemma

$\mu_{CZ}(\gamma_c) = \mu_{RS}(\Psi_{K_E}) + \mu_{RS}(\Psi_{L_3})$.

1. $\{E, L_3\} = 0. \Rightarrow dFl^H = dFl^E \circ dFl^{L_3}. \Rightarrow \Psi_H(t) = \Psi_E(t)\Psi_{L_3}(t)$.
2. K_E-flow is parallel to E-flow. $\Rightarrow \mu_{RS}(\Psi_{K_E}) = \mu_{RS}(\Psi_E)$.
3. L_3-flow is constant along $\gamma_{c_\pm}$, so the trivialization doesn't matter.

Conley-Zehnder Index of Vertical Collision Orbits

2. Linearized Flow of K_E

Parametrization of vertical collision orbits $\gamma_{c_\pm}$

$$\gamma_{c_\pm} = (-\cos(rt), 0, 0, \mp\sin(rt)\,;\, \sin(rt)/r, 0, 0, \mp\cos(rt)/r)$$

Symplectic frame along $\gamma_{c_\pm}$

$$(X_1, X_2, X_3, X_4) = (\partial_{y_1}, \partial_{x_1}, \partial_{y_2}, \partial_{x_2})$$

Hamiltonian equation of K_E

$$\begin{pmatrix} \dot{y}_1 \\ \dot{x}_1 \\ \dot{y}_2 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} |y|^2 f(-y_2(1-x_0) - x_1(r + (x_1y_2 - x_2y_1))) \\ f^2 y_1 - |y|^2 f x_1 x_2 (1-x_0) \\ |y|^2 f(ay_1(1-x_0) - x_2(r + (x_1y_2 - x_2y_1))) \\ f^2 y_2 + |y|^2 f x_1 x_2 (1-x_0) \end{pmatrix}$$

where $f(x,y) = r + (1-x_0)(x_1y_2 - x_2y_1)$

Conley-Zehnder Index of Vertical Collision Orbits

2. Linearized Flow of K_E

Imposing $x_1 = x_2 = y_1 = y_2 = 0$ (vertical),

linearized Hamiltonian flow is

$$\mathbf{L} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & r^2 & 0 \end{pmatrix}$$

By integration, we get

$$\Psi_{K_E}(t) = \begin{pmatrix} \cos(rt) & \sin(rt)/r & 0 & 0 \\ -r\sin(rt) & \cos(rt) & 0 & 0 \\ 0 & 0 & \cos(rt) & \sin(rt)/r \\ 0 & 0 & -r\sin(rt) & \cos(rt) \end{pmatrix}$$

$\Rightarrow$ A crossing at each endpoint of $\gamma_{c_\pm}$.

Conley-Zehnder Index of Vertical Collision Orbits

3. Crossing Form of K_E

Crossing form has signature 4,

$$\Omega\dot{\Psi}_{K_E}(\tau) = \begin{pmatrix} r^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\Rightarrow$ One iteration increases the index by 4.

$\Rightarrow$ $\mu_{CZ}(\gamma^N) = 4N$ for the collision orbit of (non-rotating) Kepler problem.

Conley-Zehnder Index of Vertical Collision Orbits

4. Index of Ψ_{L_3}

Linearized L_3-flow in the given basis is

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Crossings are at $2\pi\mathbb{Z}$, and the crossing form is

$$\Omega\dot{\Psi}_L(\tau) = \Omega\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The signature is 0.

Note. Fl^{L_3} rotates Lagrangian subspaces, q_1q_2-plane and p_1p_2-plane.

Conley-Zehnder Index of Vertical Collision Orbits

5. The Formula

Theorem

Let $\gamma_{c_\pm}$ be the vertical collision orbits of Kepler energy E where $E \neq E_{k,l}$ for any k, l. Then $\gamma_{c_\pm}$ and their multiple covers are non-degenerate. The Conley-Zehnder index of N-th iteration of $\gamma_{c_\pm}$ is

$$\mu_{CZ}(\gamma_{c_\pm}^N) = \mu_{RS}(\Psi_{K_E}) + \mu_{RS}(\Psi_{L_3}) = 4N + 0 = 4N.$$

Interpretation by Symplectic Homology

$$SH_*^{+,S^1}(T^*S^3) \simeq \begin{cases} \mathbb{Z}_2 & * = 2, \\ \mathbb{Z}_2^2 & * = 2k \geq 4, \\ 0 & \text{otherwise.} \end{cases}$$

For fixed N, there exists $c \ll -3/2$ such that $H^{-1}(c)$ consists of

1. $k(\leq N)$-th covers of retrograde orbit of index $4k-2$
2. $k(\leq N)$-th covers of direct orbit of index $4k+2$
3. $k(\leq N)$-th covers of vertical collision orbits of index $4k$
4. Higher covers have degree $> 4N+2$.

$\Rightarrow$ Up to degree $4N+2$, we have

1. One generator at degree 2. (γ_+.)
2. Two generators at degree $6,\ 10,\ 14,\ \cdots,\ 4N+2$. ($\gamma_+^{k+1}$ and γ_-^k.)
3. Two generators at degree $4,\ 8,\ 12,\ \cdots,\ 4N$. ($\gamma_{c_+}^k$ and $\gamma_{c_-}^k$.)

$\Rightarrow$ Describes $SH_*^{+,S^1}(T^*S^3)$ up to degree $4N+2$ completely.

Conley-Zehnder Index of Degenerate Orbits

As we increase the Kepler energy level E from $E_{k,l} - \varepsilon$ to $E_{k,l} + \varepsilon$,

1. **Retrograde**: $\mu_{CZ}(\gamma_+^{k+l})$ decreases from $4k+2$ to $4k-2$.
2. **Direct**: $\mu_{CZ}(\gamma_-^{k-l})$ increases from $4k-2$ to $4k+2$.
3. **Elliptic Orbits**: At $E = E_{k,l}$, S^3-family of orbits emerges.

Claim. Index of S^3-family of orbits with Kepler energy $E_{k,l}$ is

$$\begin{aligned}\mu_{CZ}(\Sigma) &= \text{shift}(\Sigma) + \dim S^3/2 \\ &= (4k-2) + 3/2 = 4k - 1/2\end{aligned}$$

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Conley-Zehnder Index of Degenerate Orbits

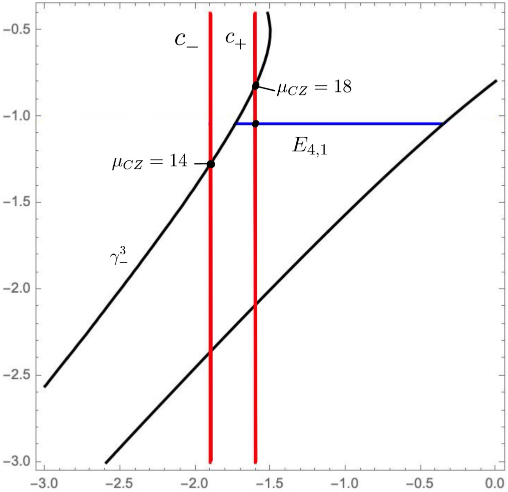
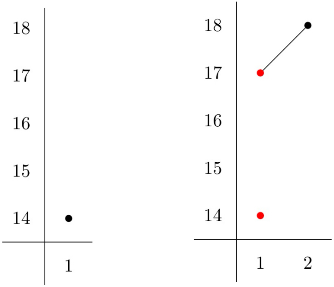


Figure 10: Index change of triple cover of direct orbit through $E_{4,1}$

Morse-Bott Spectral Sequence

(Local) Morse-Bott spectral sequence of $SH^{S^1,+}$



Left: $H = c_-$, triple cover of direct orbit with index 14.

Right: $H = c_+$, triple cover of direct orbit with index 18,

$\Rightarrow S^3$-family must have shift 14, so $\mu_{CZ}(\Sigma) = 14 + 3/2 = 15.5$.

Closing

Further Discussions

1. Showing that the S^3-family of orbits is Morse-Bott.
2. Using the result to three-body problem,
   regarded as a perturbation of Kepler problem.
3. Existence of the closed spatial orbit of three-body problem.
   (We have numerical results.)